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ATTITUDE INTERPOLATION

Sergei Tanygin*

Interpolation may be required for attitude data when re-propagation is either impossible or impractical. Nonlinearity of kinematics and potential aliasing between revolutions are among the unique challenges presented by the attitude motion. Selections of the reference frame and attitude parameterization become very important. It is possible to apply standard polynomial interpolation techniques to the attitude data and, also, include the angular velocity data in order to achieve a better accuracy and to reduce potential for aliasing between revolutions. It is also possible to employ interpolated polynomial trajectories for fixed duration near optimal maneuver design and to achieve additional optimization for spinners.

INTRODUCTION

A need for interpolation may generally arise given a set of grid points, where each point contains corresponding values of independent and dependent variables. Interpolation is one of the ways to produce values between the points in the absence of other information relating dependent and independent variables. Interpolation also does that while ensuring that values at the grid points are matched. Interpolation methods vary according to what type of functions they use, how many grid points they use and how many derivatives they can take advantage of at each point. One of the simplest and most common types of functions used for interpolation is a polynomial. Table 1 includes a classification of polynomial interpolation methods according to the number of grid points and derivatives.[1]

TABLE 1 POLYNOMIAL INTERPOLATION METHODS

	Two points	*>Two points*
No derivatives	*Linear Lagrange*	*Lagrange*
1st derivatives	*1st Order Taylor or 2-point Osculating*	*Osculating*
1st and higher derivatives	*Taylor*	*Hermite*

* Member AAS and AIAA. Lead Engineer, Analytical Graphics, Inc., 40 General Warren Blvd., Malvern, PA 19355, stanygin@stk.com

In practice, ephemeris interpolation has been used extensively when position and optionally velocity data are available at a discrete set of times. The data may originate from simulations or from telemetry and estimation. Re-propagation of ephemeris can be an alternative to interpolation, but it requires knowledge of force models. Even if force models are known, the amount of extra data and computations may become unnecessary when desired accuracy can be satisfied with interpolation. Conceptually, attitude interpolation parallels ephemeris interpolation: it enables relatively fast and accurate computation of attitude and angular velocity at any time spanned by a discrete set of points. The differences come from non-linear nature of attitude composition operations and attitude kinematics. Results of interpolation will depend on both attitude parameterization and reference frame selected for data points. The most straightforward approach calls for independent interpolation of each element of parameterization. At the same time, this places significant restrictions on types of acceptable parameterizations: attitude parameterization becomes unsuitable for interpolation if it imposes constraints on its elements or if it exhibits discontinuities or singularities.

TABLE 2 ATTITUDE PARAMETERIZATIONS

	Constraints	*Singularities*	*Discontinuities*
Direction cosine matrix	X		
Unit quaternion	X		
Cayley-Klein parameters	X		
3 subsequent angle-axis rotations		X	
Eigen-axis and function of eigen-angle		X	X
• ***Rotation vector***		X	

According to Table 2, these restrictions eliminate from the consideration every possible parameterization. However, a closer examination of the rotation vector parameterization reveals that its singularity at the origin can be easily resolved. The general definition and rotation vector kinematics are described below:[2]

$$\mathbf{q} = \begin{bmatrix} \hat{\boldsymbol{\varphi}} \sin\phi / 2 \\ \cos\phi / 2 \end{bmatrix}, \tag{1}$$

$$\begin{aligned} \boldsymbol{\omega} &= [1 - \sin\phi / \phi](\hat{\boldsymbol{\varphi}}^{\mathbf{T}} \dot{\boldsymbol{\varphi}})\hat{\boldsymbol{\varphi}} \\ &\quad + [\sin\phi / \phi]\dot{\boldsymbol{\varphi}} + [(\cos\phi - 1)/\phi](\hat{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\varphi}}), \end{aligned} \tag{2}$$

$$\begin{aligned} \dot{\boldsymbol{\omega}} &= \ddot{\boldsymbol{\varphi}} - (\dot{\phi}/\phi)[\sin\phi/\phi - 2(1-\cos\phi)/\phi^2](\hat{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\varphi}}) \\ &\quad - [(1-\cos\phi)/\phi](\hat{\boldsymbol{\varphi}} \times \ddot{\boldsymbol{\varphi}}) \\ &\quad + (\dot{\phi}/\phi)[(1-\cos\phi) - 3(1-\sin\phi/\phi)]\hat{\boldsymbol{\varphi}} \times (\hat{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\varphi}}), \\ &\quad + [(\phi - \sin\phi)/\phi^2][\phi\hat{\boldsymbol{\varphi}} \times (\hat{\boldsymbol{\varphi}} \times \ddot{\boldsymbol{\varphi}}) + \dot{\boldsymbol{\varphi}} \times (\hat{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\varphi}})] \end{aligned} \tag{3}$$

where $\mathbf{q}$ is the four-parameter vector representing the attitude in terms of the unit quaternion, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the body angular velocity and acceleration in the body fixed frame, $\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}}\phi$ is the rotation vector with the direction $\hat{\boldsymbol{\varphi}}$ along the eigen-axis of rotation relative to the reference frame and with the magnitude ϕ equal to the eigen-angle of rotation. However, near the origin, the definition is simplified to

$$\mathbf{q} \to \begin{bmatrix} \boldsymbol{\varphi}/2 \\ 0 \end{bmatrix} \tag{4}$$

and the kinematical relationships reduce to direct correspondence of the rotation vector velocity to the body angular velocity and the rotation vector acceleration to the body angular acceleration:

$$\boldsymbol{\omega} \to \dot{\boldsymbol{\varphi}}, \tag{5}$$

$$\dot{\boldsymbol{\omega}} \to \ddot{\boldsymbol{\varphi}}. \tag{6}$$

These results are important not only for interpolation, but also for design of near optimal attitude maneuvers later in this paper. Finally, note that norm of the rotation vector corresponds to the eigen angle, which, in turn, represents a “distance” measure in attitude space.

ROTATION VECTOR INTERPOLATION

A method for attitude interpolation using rotation vector parameterization is presented in this section. The method combines useful properties of this parameterization described in the introduction with standard polynomial interpolation techniques. The method contains the following steps:

1. Find N attitude grid points centered around time of interest
2. Redefine these N points with respect to attitude of grid point nearest in time
3. Convert resulting points to rotation vector parameterization
4. Perform N point Lagrange interpolation on each rotation vector element independently
5. Convert resulting rotation vector to desired attitude parameterization

The procedure outlined above does not require knowledge of the angular velocity: Step 4 is performed for the attitude alone. The angular velocity can be interpolated separately by also using Lagrange interpolation provided that angular velocity grid points are available. However, de-coupling the attitude from the angular velocity ignores kinematical relationship between the two and may result in a significant loss of accuracy. Note that interpolation of rotations presents a special challenge: without the angular velocity data it may not be possible for the interpolation to distinguish between attitudes separated by complete revolutions, the effect often referred to as aliasing of revolutions. This effect may ultimately result in a sign error causing the apparent interpolated motion to proceed in the direction opposite to the actual motion. These problems provide strong arguments for incorporating the angular velocity data into the attitude interpolation. This can be done by replacing the two separate Lagrange interpolations, one for the attitude and the other for the angular velocity, with the single osculating interpolation that operates on the rotation vector and its first derivative. The derivative needs to be related to the angular velocity via rotation vector kinematics (Eq.(2)). Hence, the interpolation method incorporating angular velocity contains the following steps:

1. Find N attitude and angular velocity points centered around time of interest
2. Redefine these N points with respect to attitude of point nearest in time
3. Convert resulting attitude and angular velocity points to rotation vector parameterization and its velocity
4. Perform N point osculating interpolation on each pair of rotation vector element and its derivative independently; each element of rotation vector velocity is interpolated using derivative of interpolating polynomial

5. Convert resulting rotation vector and its velocity to desired attitude parameterization and angular velocity

Note that N-point Lagrange interpolation employs polynomials of degree N-1, whereas N-point osculating interpolation employs polynomials of degree 2N-1. It is instructive to consider 2-point interpolations in more details in order to gain a better insight into the construction of interpolating polynomials as well as in order to establish the mathematical foundation for the next section.

2-point osculating interpolation employs cubic polynomial that passes between two grid points leaving and arriving at specified slopes in specified time (Fig. 1). Without loss of generality, the time of the first grid point can be set to 0, so that the second grid point simply occurs at the time elapsed between the two points, *T*, and so that the time along the polynomial progresses between 0 and *T*.

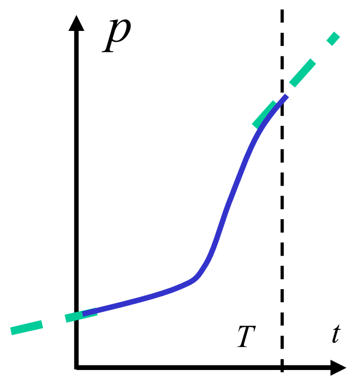

Figure 1 Cubic polynomial *p(t)* in 2-point osculating interpolation

2-point osculating cubic polynomial is constructed as a linear combination of four other cubic polynomials. These polynomials, often referred to as basis, have coefficients that depend only on the time elapsed between the grid points and do not depend on either values or slopes at the two grid points. Each of the four basis polynomials is then multiplied by one of four constants: two values and two slopes at the grid points. These four scaled polynomials added together compose the 2-point osculating cubic polynomial. Hence, the interpolation of the rotation vector $\boldsymbol{\varphi}(t)$ and its derivatives $\dot{\boldsymbol{\varphi}}(t)$, $\ddot{\boldsymbol{\varphi}}(t)$ results in cubic, quadratic and linear polynomials, $\overline{\boldsymbol{\varphi}}(t)$, $\dot{\overline{\boldsymbol{\varphi}}}(t)$ and $\ddot{\overline{\boldsymbol{\varphi}}}(t)$, respectively. All of them can be formulated as a linear combination of the four other polynomials in the vector form:

$$\overline{\boldsymbol{\varphi}}(t) = \boldsymbol{\varphi}_{\mathbf{0}}\overline{p}_0(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{0}}\overline{r}_0(t) + \boldsymbol{\varphi}_{\mathbf{T}}\overline{p}_T(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{T}}\overline{r}_T(t)\,, \tag{7}$$

$$\dot{\overline{\boldsymbol{\varphi}}}(t) = \boldsymbol{\varphi}_{\mathbf{0}}\dot{\overline{p}}_0(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{0}}\dot{\overline{r}}_0(t) + \boldsymbol{\varphi}_{\mathbf{T}}\dot{\overline{p}}_T(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{T}}\dot{\overline{r}}_T(t)\,, \tag{8}$$

$$\ddot{\overline{\boldsymbol{\varphi}}}(t) = \boldsymbol{\varphi}_{\mathbf{0}}\ddot{\overline{p}}_0(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{0}}\ddot{\overline{r}}_0(t) + \boldsymbol{\varphi}_{\mathbf{T}}\ddot{\overline{p}}_T(t) + \dot{\boldsymbol{\varphi}}_{\mathbf{T}}\ddot{\overline{r}}_T(t)\,, \tag{9}$$

where $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_{\mathbf{0}}$, $\dot{\boldsymbol{\varphi}}(0) = \dot{\boldsymbol{\varphi}}_{\mathbf{0}}$, $\boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_{\mathbf{T}}$, $\dot{\boldsymbol{\varphi}}(T) = \dot{\boldsymbol{\varphi}}_{\mathbf{T}}$ and

$$\bar{p}_0(t) = \left[1 + 2\frac{t}{T}\right]\frac{(t-T)^2}{T^2}, \tag{10}$$

$$\bar{p}_T(t) = 1 - \bar{p}_0(t) = \left[3 - 2\frac{t}{T}\right]\frac{t^2}{T^2}, \tag{11}$$

$$\bar{r}_0(t) = \frac{t(t-T)^2}{T^2}, \tag{12}$$

$$\bar{r}_T(t) = \frac{(t-T)t^2}{T^2}, \tag{13}$$

$$\dot{\bar{p}}_0(t) = -\dot{\bar{p}}_T(t) = 6\frac{t(t-T)}{T^3}, \tag{14}$$

$$\dot{\bar{r}}_0(t) = \frac{(t-T)(3t-T)}{T^2}, \tag{15}$$

$$\dot{\bar{r}}_0(t) = \frac{t(3t-2T)}{T^2}, \tag{16}$$

$$\ddot{\bar{p}}_0(t) = -\ddot{\bar{p}}_T(t) = 6\frac{2t-T}{T^3}, \tag{17}$$

$$\ddot{\bar{r}}_0(t) = 2\frac{3t-2T}{T^2}, \tag{18}$$

$$\ddot{\bar{r}}_T(t) = 2\frac{3t-T}{T^2}. \tag{19}$$

As stated above, the essential properties of 2-point osculating interpolation are $\bar{\boldsymbol{\varphi}}(0) = \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_{\mathbf{0}}$, $\dot{\bar{\boldsymbol{\varphi}}}(0) = \dot{\boldsymbol{\varphi}}(0) = \dot{\boldsymbol{\varphi}}_{\mathbf{0}}$, $\bar{\boldsymbol{\varphi}}(T) = \boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_{\mathbf{T}}$, $\dot{\bar{\boldsymbol{\varphi}}}(T) = \dot{\boldsymbol{\varphi}}(T) = \dot{\boldsymbol{\varphi}}_{\mathbf{T}}$. In turn, these properties can be deduced from observing the following properties of the basis polynomials:

$$\bar{p}_0(0) = \bar{p}_T(T) = \dot{\bar{r}}_0(0) = \dot{\bar{r}}_T(T) = 1, \tag{20}$$

$$\begin{aligned} \bar{p}_0(T) &= \bar{p}_T(0) = \bar{r}_0(0) = \bar{r}_T(0) = \bar{r}_0(T) = \bar{r}_T(T) \\ &= \dot{\bar{p}}_0(0) = \dot{\bar{p}}_T(0) = \dot{\bar{p}}_0(T) = \dot{\bar{p}}_T(T) = 0 \end{aligned}. \tag{21}$$

The construction of 2-point osculating polynomials can be somewhat simplified if the attitude of one of the grid points is used as the reference frame. For example, re-defining both grid points to be relative to the attitude of the first point, makes that point the origin and makes its rotation vector parameterization a zero vector, $\boldsymbol{\varphi}_{\mathbf{0}} = \mathbf{0}$. Thus, 2-point osculating polynomials become a linear combination of only three basis polynomials. Another benefit of placing the origin at one of the grid points is the simplification of kinematical relationships of the rotation vector velocity and acceleration with angular velocity and acceleration: near the origin, the the rotation vector velocity and acceleration approximately become the angular velocity and acceleration, respectively (Eqs.(5,6)).

This and previous sections presented the attitude interpolation methods that utilize the attitude and, optionally, the angular velocity data. The angular acceleration along the interpolating polynomials was also formulated here, but its significance is the subject of the next section.

NEAR OPTIMAL FIXED DURATION MANEUVER

Given an attitude maneuver with specific duration and specific initial and target attitudes (and angular velocities), minimizing the overall torque spent during the maneuver is certainly one of most desirable objectives. The challenge lies in relating the shape of the attitude trajectory to the torque along that trajectory: both kinematics and dynamics must be accounted for and they are generally non-linear. Hence, one reason to be interested in the angular acceleration along the interpolating polynomial is because it relates to the rotation vector acceleration on one side and to the body fixed torque on the other side. The 2-point osculating polynomial using rotation vector parameterization quickly emerges as a good candidate for the desired trajectory because of its two properties:

1. Trajectory passes between initial and final attitudes leaving and arriving with specified angular velocities in specified time

2. Trajectory is least curved in rotation vector parameterization

The first property was fully discussed in the previous section. The second property comes from using calculus of variations to minimize the following objective functions:

$$J(\phi_i) = \int_0^T \ddot{\phi}_i^2 dt, i = 1,2,3, \tag{22}$$

where $\phi_i \in \Re^1$ is the ith element of the rotation vector $\boldsymbol{\varphi}$. The Euler-Lagrange equation[4,5] yields the following condition for ϕ_i:

$$\ddot{\ddot{\phi}}_i \equiv 0, i = 1,2,3, \tag{23}$$

which is clearly satisfied if ϕ_i is any cubic polynomial.[6] Hence, the 2-point osculating polynomials $\overline{\phi}_i$ developed in the previous section as part of $\overline{\boldsymbol{\varphi}} = \begin{bmatrix} \overline{\phi}_1 & \overline{\phi}_2 & \overline{\phi}_3 \end{bmatrix}^T$ minimize the objective functions (Eq.(22)):

$$\min_{\phi_i} J(\phi_i) = J(\overline{\phi}_i), i = 1,2,3. \tag{24}$$

This is the first step towards minimization of the ultimate objective function:

$$J(\mathbf{M}) = \int_0^T \mathbf{M}^{\mathrm{T}} \mathbf{M} dt , \tag{25}$$

where $\mathbf{M} \in \Re^3$ is the applied torque.

The results below follow directly from the properties of rotation vector kinematics presented in the previous sections (Eqs.(5,6)):

$$\lim_{\boldsymbol{\varphi} \to 0} \ddot{\boldsymbol{\varphi}} = \dot{\boldsymbol{\omega}} , \tag{26}$$

$$\lim_{\dot{\boldsymbol{\varphi}} \to 0} \ddot{\boldsymbol{\varphi}} = \dot{\boldsymbol{\omega}} , \tag{27}$$

$$\lim_{\substack{\angle(\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}) \to 0 \\ \angle(\boldsymbol{\varphi}, \ddot{\boldsymbol{\varphi}}) \to 0}} \ddot{\boldsymbol{\varphi}} = \dot{\boldsymbol{\omega}} . \tag{28}$$

In other words, the rotation vector acceleration approaches the angular acceleration under either one of the following three conditions:

1. Trajectory is small
2. Trajectory is slow
3. Trajectory is close to maintained pure spin

This also means that the objective functions based on the angular acceleration components are approximately minimized under the same conditions:

$$\min_{\dot{\omega}_i} \int_0^T \dot{\omega}_i^2 dt \approx \int_0^T \dot{\overline{\omega}}_i^2 dt, i = 1,2,3 , \tag{29}$$

where $\dot{\omega}_i, \dot{\overline{\omega}}_i \in \Re^1$ and where $\dot{\overline{\omega}}_i$ is computed as part of the angular acceleration vector $\dot{\overline{\boldsymbol{\omega}}} = \begin{bmatrix} \dot{\overline{\omega}}_1 & \dot{\overline{\omega}}_2 & \dot{\overline{\omega}}_3 \end{bmatrix}^{\mathrm{T}}$ along the 2-point osculating rotation vector trajectory.

The next step towards minimization of the applied torque considers the time rate of change of the rigid body angular momentum in the body fixed frame, $\mathbf{I}\dot{\boldsymbol{\omega}} \in \Re^3$, where $0 < \mathbf{I}^{\mathrm{T}} = \mathbf{I} \in \Re^{3 \times 3}$ is the body fixed inertia matrix. The following relationship can be established between the objective functions dealing with every component of the angular acceleration, $\dot{\omega}_i$, and the objective function dealing with $\mathbf{I}\dot{\boldsymbol{\omega}}$:

$$\min_{\dot{\omega}_i} \int_0^T \dot{\omega}_i^2 dt, i = 1,2,3 \Rightarrow \min_{\dot{\omega}_i, i=1,2,3} \int_0^T (\mathbf{I}\dot{\boldsymbol{\omega}})^{\mathrm{T}} \mathbf{I}\dot{\boldsymbol{\omega}} dt , \tag{30}$$

where the relationship is, of course, preserved in any body fixed frame, but is particularly evident in the principal frame.

The final step must include attitude dynamics in order to relate the time rate of change of the rigid body angular momentum in the body frame to the applied torque:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{M}\,, \tag{31}$$

where $\mathbf{H} \in \Re^3$ is the body total angular momentum in the body fixed frame, which includes the angular momentum of the rigid body itself and the internal angular momentum due to parts moving with respect to the rigid body. The effect of cross coupling, $\boldsymbol{\omega} \times \mathbf{H}$, can be negligible under either one of these two conditions:

1. Trajectory is slow
2. Trajectory is close to pure spin about one of principal axes and internal angular momentum, if any, is close to same axis

This means that under these conditions, the ultimate objective function (Eq.(25)) is approximately minimized whenever $J(\mathbf{I}\dot{\boldsymbol{\omega}}) = \int_0^T (\mathbf{I}\dot{\boldsymbol{\omega}})^{\mathrm{T}}\, \mathbf{I}\dot{\boldsymbol{\omega}}\,dt$ is minimized.

In summary, under the sets of kinematical and dynamical conditions, the maneuver trajectory defined by the 2-point osculating polynomial in the rotation vector parameterization approximately minimizes the overall magnitude of the torque applied during the maneuver. While these conditions are satisfied for a large group of maneuvers, e.g. spin-up/-down, slow large angle, etc., the most challenging agile large angle maneuvers may become far from optimal when using the proposed trajectory. The rest of this section presents trajectory modifications that may improve its optimality even for agile large angle maneuvers.

The conditions imposed by the attitude dynamics are especially difficult to relax, because of the significant and highly non-linear contribution of the cross coupling during agile maneuvers. However, because of this difficulty, the cross coupling is often compensated for agile spacecraft via the closed-loop feedback linearization in order to ensure a better predictability of attitude trajectories. Accepting this penalty on the applied torque leaves only the kinematical conditions to be relaxed. The attitude trajectories that minimize $J(\mathbf{I}\dot{\boldsymbol{\omega}})$ will be referred to as kinematically optimal in the rest of this paper.

Consider the maneuver trajectory defined in the rotation vector parameterization relative to the attitude of the initial point using the 2-point osculating interpolation. As stated in this and previous sections, this selection of the reference frame makes the trajectory depart from the origin, which means that, at this point, the rotation vector velocity and acceleration are equal to the angular velocity and acceleration, respectively. If this equivalence were maintained throughout the trajectory, the trajectory would be kinematically optimal, but, generally, this is not the case. Generally, the linear progression of the rotation vector acceleration throughout the trajectory does not correspond to the linear progression of the angular acceleration. The latter can become more and more divergent and curved as the trajectory moves further away from the origin, because the effect of kinematical non-linearity can become more and more pronounced. One of the most intuitive approaches to countering this effect is to somehow introduce periodic corrections to the interpolation polynomials. For example, while the same target attitude and angular velocity are sought, the entire interpolation problem can be re-cast using the current trajectory point as the initial point and using the remaining maneuver time to update interpolation polynomials (Fig. 2). This procedure can be repeated as often as necessary depending on the significance of the non-linear kinematical effect. The method of periodic corrections becomes effectively a closed-loop guidance method with periodic updates. For example, the rotation vector, its velocity and acceleration during the period of $0 \le t \le t' \le T$ are governed by the following set of equations:

$$\overline{\boldsymbol{\varphi}}(t) = \boldsymbol{\omega_0}\overline{r}_0(t) + \boldsymbol{\varphi_T}\overline{p}_T(t) + \dot{\boldsymbol{\varphi}}_\mathbf{T}\overline{r}_T(t)\,, \tag{32}$$

$$\dot{\overline{\boldsymbol{\varphi}}}(t) = \boldsymbol{\omega_0}\dot{\overline{r}}_0(t) + \boldsymbol{\varphi_T}\dot{\overline{p}}_T(t) + \dot{\boldsymbol{\varphi}}_\mathbf{T}\dot{\overline{r}}_T(t)\,, \tag{33}$$

$$\ddot{\overline{\boldsymbol{\varphi}}}(t) = \boldsymbol{\omega_0}\ddot{\overline{r}}_0(t) + \boldsymbol{\varphi_T}\ddot{\overline{p}}_T(t) + \dot{\boldsymbol{\varphi}}_\mathbf{T}\ddot{\overline{r}}_T(t)\,, \tag{34}$$

where all terms multiplied by $\boldsymbol{\varphi_0} \equiv 0$ are removed and where $\dot{\boldsymbol{\varphi}}_\mathbf{0}$ is replaced with $\boldsymbol{\omega_0}$, because the two are equal at the origin. At time t' the problem can be re-cast: the target rotation vector and its velocity are redefined relative to $\overline{\boldsymbol{\varphi}}(t')$ resulting in $\boldsymbol{\varphi}'_\mathbf{T}$ and $\dot{\boldsymbol{\varphi}}'_\mathbf{T}$, respectively; the basis polynomials are rebuilt using the remaining maneuver time of $T - t'$ instead of T (all updated polynomials are indicated by “prime” in the next set of equations); the original angular velocity $\boldsymbol{\omega_0}$ is replaced with its current counterpart, $\boldsymbol{\omega}'_\mathbf{0} = \boldsymbol{\omega}'(t')$. Hence, the following set of equations governs the rotation vector, its velocity and acceleration after this correction and until the end of the maneuver or until the next correction:

$$\overline{\boldsymbol{\varphi}}'(t) = \boldsymbol{\omega}'_\mathbf{0}\overline{r}'_0(t) + \boldsymbol{\varphi}'_\mathbf{T}\overline{p}'_T(t) + \dot{\boldsymbol{\varphi}}'_\mathbf{T}\overline{r}'_T(t)\,, \tag{35}$$

$$\dot{\overline{\boldsymbol{\varphi}}}'(t) = \boldsymbol{\omega}'_\mathbf{0}\dot{\overline{r}}'_0(t) + \boldsymbol{\varphi}'_\mathbf{T}\dot{\overline{p}}'_T(t) + \dot{\boldsymbol{\varphi}}'_\mathbf{T}\dot{\overline{r}}'_T(t)\,, \tag{36}$$

$$\ddot{\overline{\boldsymbol{\varphi}}}'(t) = \boldsymbol{\omega}'_\mathbf{0}\ddot{\overline{r}}'_0(t) + \boldsymbol{\varphi}'_\mathbf{T}\ddot{\overline{p}}'_T(t) + \dot{\boldsymbol{\varphi}}'_\mathbf{T}\ddot{\overline{r}}'_T(t)\,. \tag{37}$$

Note that the time t in the equations above is also reset: it starts at 0 and, in the absence of other corrections, continues until the end of the maneuver at time $T - t'$.

Recall that, in general, more frequent corrections result in smaller contributions from the non-linear kinematics and, thus, result in less curved angular velocity and acceleration trajectories. Therefore, it is natural to seek a transition from periodic corrections to continuous correction throughout the trajectory. The transition becomes clear if the angular acceleration at the initial point of any of the periodic corrections is examined:

$$\dot{\bar{\boldsymbol{\omega}}}'(0) = \ddot{\bar{\boldsymbol{\varphi}}}'(0) = -\boldsymbol{\omega}_0' \frac{4}{T-t'} + \boldsymbol{\varphi}_\mathbf{T}' \frac{6}{(T-t')^2} - \dot{\boldsymbol{\varphi}}_\mathbf{T}' \frac{2}{T-t'} . \tag{38}$$

This equation can be rewritten as a continuous function of the original time t assuming that the trajectory includes an infinite number of corrections, each applied over an infinitely small period of time. In other words, any point on this trajectory can be considered the initial point of some correction. The resulting equation after straightforward re-grouping takes form of the closed-loop guidance law. The law specifies the angular acceleration at any point on the trajectory as a function of the current time, total maneuver duration, the current attitude and angular velocity as well as the target attitude and angular velocity:

$$\dot{\bar{\boldsymbol{\omega}}}(t) = \ddot{\bar{\boldsymbol{\varphi}}}(t) = 2\frac{3\boldsymbol{\varphi}_\mathbf{T}(t) - [2\boldsymbol{\omega}(t) + \dot{\boldsymbol{\varphi}}_\mathbf{T}(t)](T-t)}{(T-t)^2} , \tag{39}$$

where $\boldsymbol{\varphi}_\mathbf{T}(t)$ and $\dot{\boldsymbol{\varphi}}_\mathbf{T}(t)$ are the target rotation vector and its velocity relative to the current attitude along the trajectory. The apparent singularity in Eq.(39) as $t \to T$ is resolved, because its numerator becomes proportional to $(T-t)^2$ just as $t \to T$, so that

$$\lim_{t \to T} \dot{\bar{\boldsymbol{\omega}}}(t) = \mathbf{const} . \tag{40}$$

In practice, as $t \to T$, the guidance law may be turned off and replaced with a simple attitude control law tracking the target attitude and angular velocity. The guidance law uses continuous local linearization and effectively “slides” interpolating polynomials along the trajectory towards the target (Fig. 2). The resulting trajectory reaches the target at the specified time in a near kinematically optimal manner.

The importance of the appropriate selection of the reference frame for interpolation and for maneuver design is clearly evident. The next section demonstrates the case when not only the initial attitude, but also the target attitude of the maneuver is utilized as the reference frame during the maneuver design.

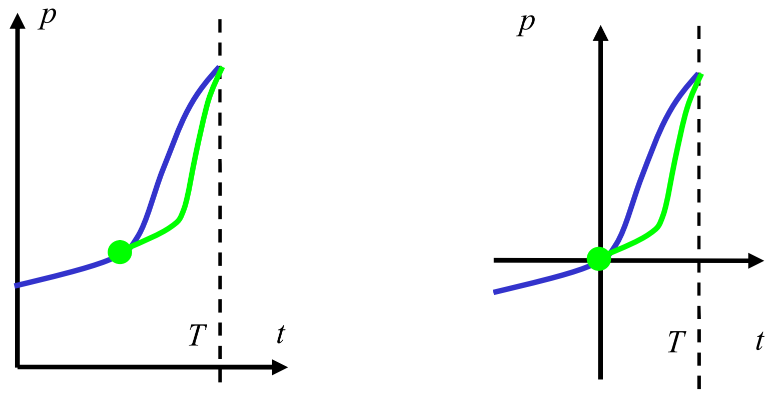

Figure 2 “Sliding” 2-point osculating interpolation of *p(t)*

CLOCK-ANGLE OPTIMIZATION FOR SPINNERS

Consider the attitude trajectory for which the initial point, the target point or both specify only the angular velocity vector and its orientation in some reference frame. For example, if the target trajectory point only specifies the angular velocity, the target attitude itself is not fully defined and possesses an additional degree of freedom: the clock-angle α_T about the angular velocity vector (Fig. 3).

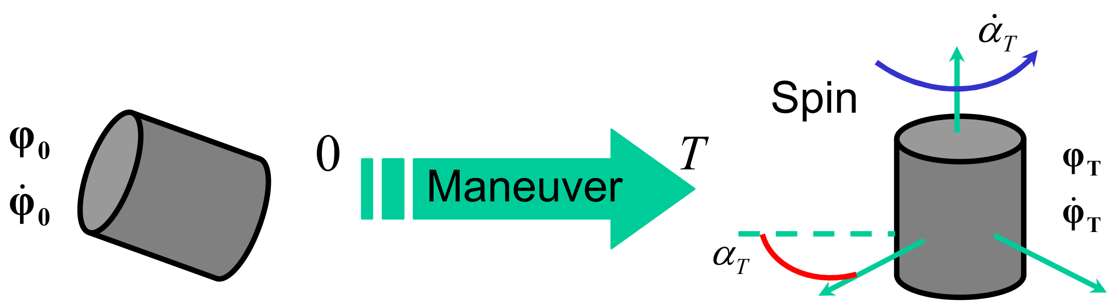

Figure 3 Target clock-angle for spinners

Additional optimization may be performed with respect to this parameter, α_T, assuming that the attitude trajectory takes the form of the 2-point osculating polynomial described in the previous sections. The manner in which the clock-angle enters the cost function strongly depends on the selection of the reference frame for the interpolated rotation vector. In general, the composition operation in the rotation vector parameterization is non-linear and includes trigonometric functions. However, the deliberate selection of the inertial reference frame that aligns one of its axes, e.g. the third axis, with the target angular velocity can linearize the composition operation and limit the clock-angle dependency to only one component of the rotation vector, e.g. $\bar{\phi}_3$ (Fig. 4).

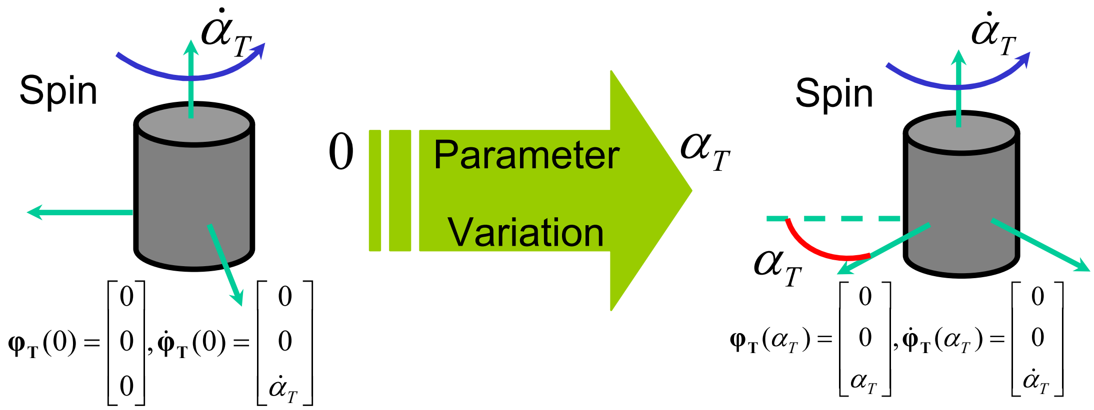

Figure 4 Clock-angle variation

Then, the resulting cost function for the parametric optimization is straightforward:

$$J_p(\alpha_T) = \int_0^T \ddot{\bar{\phi}}_3^2(\alpha_T)dt \tag{41}$$

and so is the solution to the parameter optimization problem[4,5] $\min_{\alpha_T} J_p(\alpha_T) = J_p(\bar{\alpha}_T)$:

$$\bar{\alpha}_T = \phi_{03} + \frac{\dot{\phi}_{03} + \dot{\alpha}_T}{2} T, \tag{42}$$

where ϕ_{03} and $\dot{\phi}_{03}$ are third components of the initial rotation vector and its velocity relative to the inertial reference frame that aligns its third axis with the target angular velocity; $\dot{\alpha}_T$ is the magnitude of the target angular velocity in that same frame. Note that in that frame both the angular velocity and the rotation vector velocity are aligned with the third axis. As expected, the optimal clock-angle value $\bar{\alpha}_T$ is the one that minimizes curvature of the rotation vector velocity, which ultimately leads to the reduction by one of the polynomial degrees for the third component of the rotation vector, its velocity and acceleration (Fig. 5):

$$\bar{\phi}_3(t) = \phi_{03} + \dot{\phi}_{03}t + \left(\dot{\alpha}_T - \dot{\phi}_{03}\right)\frac{t^2}{2T}, \tag{43}$$

$$\dot{\bar{\phi}}_3(t) = \dot{\phi}_{03} + \left(\dot{\alpha}_T - \dot{\phi}_{03}\right)\frac{t}{T}, \tag{44}$$

$$\ddot{\bar{\phi}}_3(t) = \frac{\dot{\alpha}_T - \dot{\phi}_{03}}{T}. \tag{45}$$

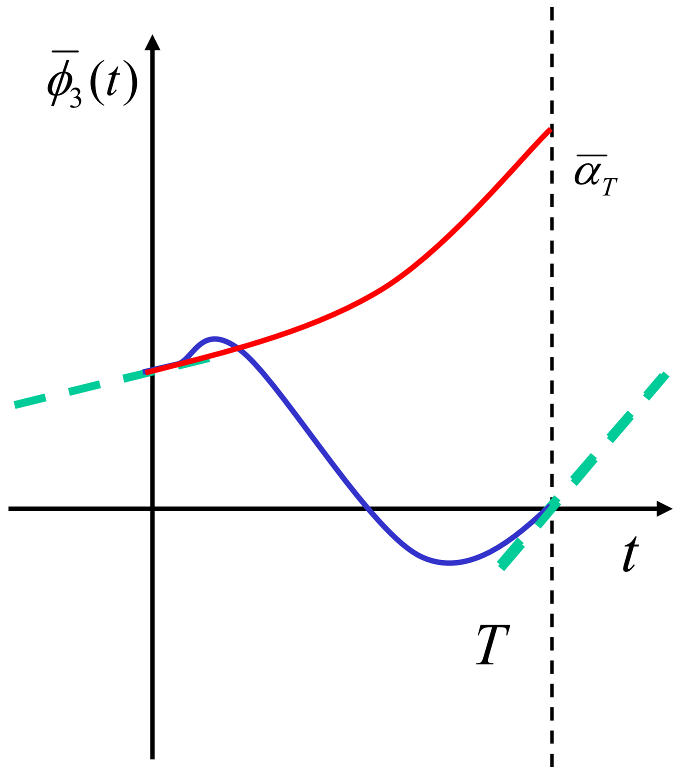

Figure 5 Reduction of curvature with optimal clock-angle

EXAMPLES

This section illustrates the performance of the closed-loop guidance law. The following conditions are defined for the maneuver:

- Initial quaternion and angular velocity:

  $\mathbf{q}(0s) = [0 \quad 0 \quad 1 \quad 0]^{\mathrm{T}}$

  $\boldsymbol{\omega}(0s) = [0 \quad 0 \quad 0.055]^{\mathrm{T}}$ deg/s

- Target quaternion and angular velocity:

  $\mathbf{q}(180s) = [0.3829 \quad 0.6621 \quad -0.4139 \quad -0.4936]^{\mathrm{T}}$

  $\boldsymbol{\omega}(180s) = [-0.314 \quad -0.6947 \quad 0.0226]^{\mathrm{T}}$ deg/s

- Agile maneuver with same target quaternion, but higher target angular velocity:

$$\boldsymbol{\omega}(180s) = \begin{bmatrix} -3.11 & -6.934 & 0.24 \end{bmatrix}^{\mathrm{T}} \text{ deg/s}$$

The kinematical performance measure related to the original cost function is defined as follows:

$$Jr(t) = \int_0^t \sqrt{\dot{\boldsymbol{\omega}}^{\mathrm{T}}\dot{\boldsymbol{\omega}}}\, d\tau \,. \tag{46}$$

The error eigen-angle is computed relative to the frame aligned with the target attitude at the final time. Similarly, the error for the rotation vector velocity $\dot{\boldsymbol{\varphi}}(t)$ is computed relative to the frame aligned and rotating with the target frame at the final time. The error for the angular velocity $\boldsymbol{\omega}(t)$ is computed as a direct difference between its current vector and the target vector. As expected, all of the errors reach zero at the final time and the difference between the errors in the rotation vector velocity and the angular velocity tend to become equivalent as the attitude trajectory approaches the target (Figs.6-13). Also, as expected, the agile maneuver exhibits additional variations in the trajectories (Figs.10-13). It is interesting to note that, despite these initial variations, half way into the maneuver, the rotation vector velocity and the angular velocity (Fig. 12) become very regular and become shaped much like those of the slow maneuver (Fig.8).

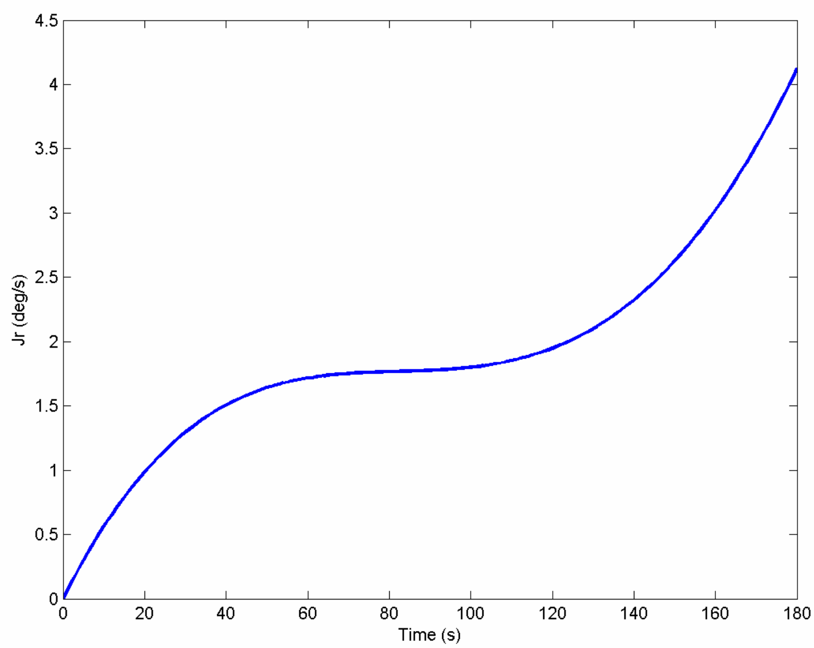

Figure 6 Performance measure

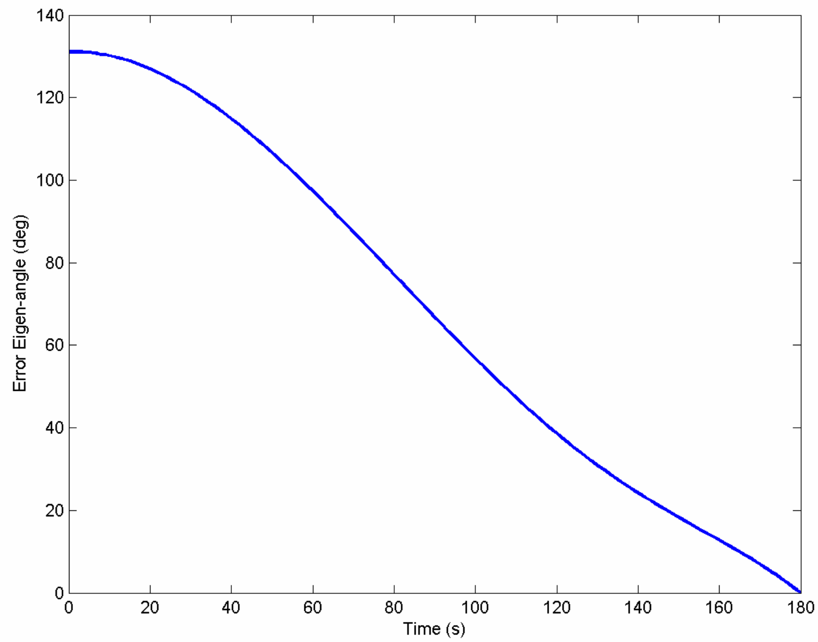

Figure 7 Error eigen-angle relative to target attitude

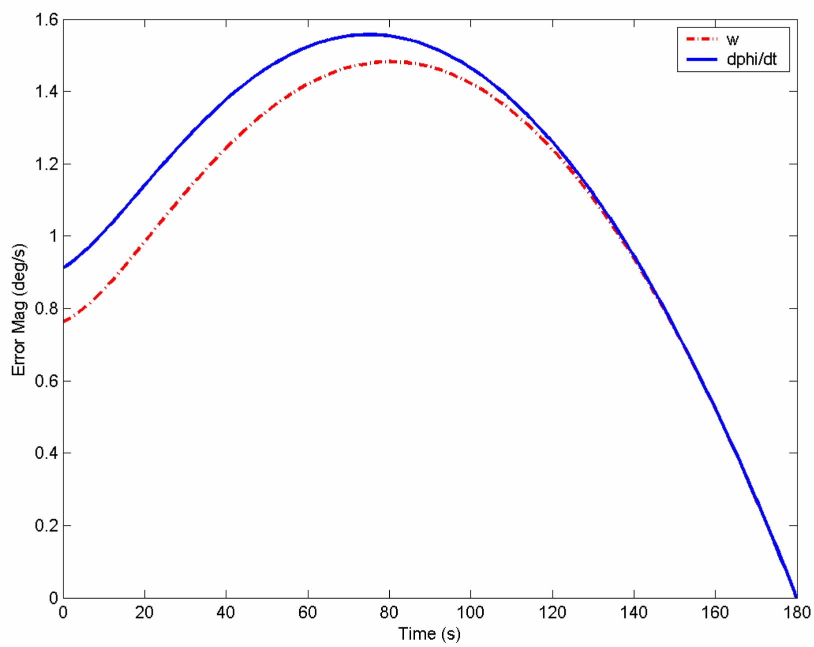

Figure 8 Error magnitudes in terms of angular velocity and rotation vector velocity relative to target attitude

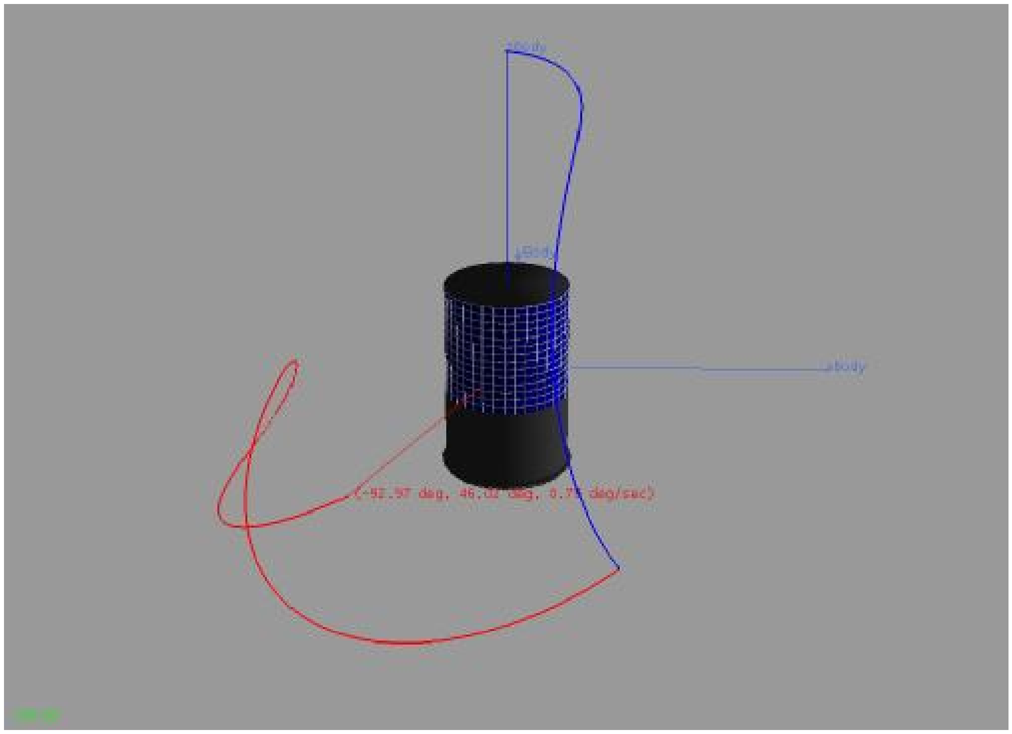

Figure 9 Motion of Z-body axis and angular velocity direction

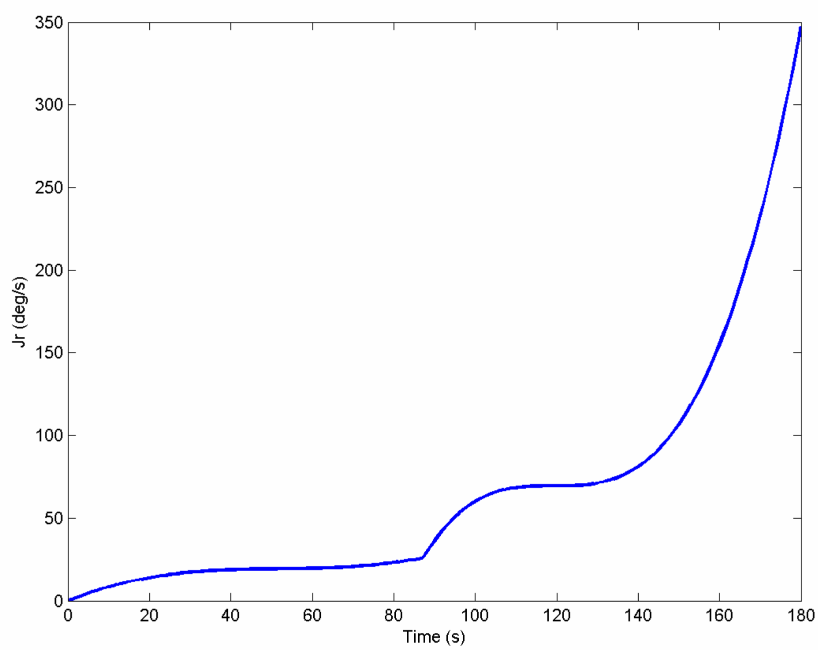

Figure 10 Agile maneuver performance measure

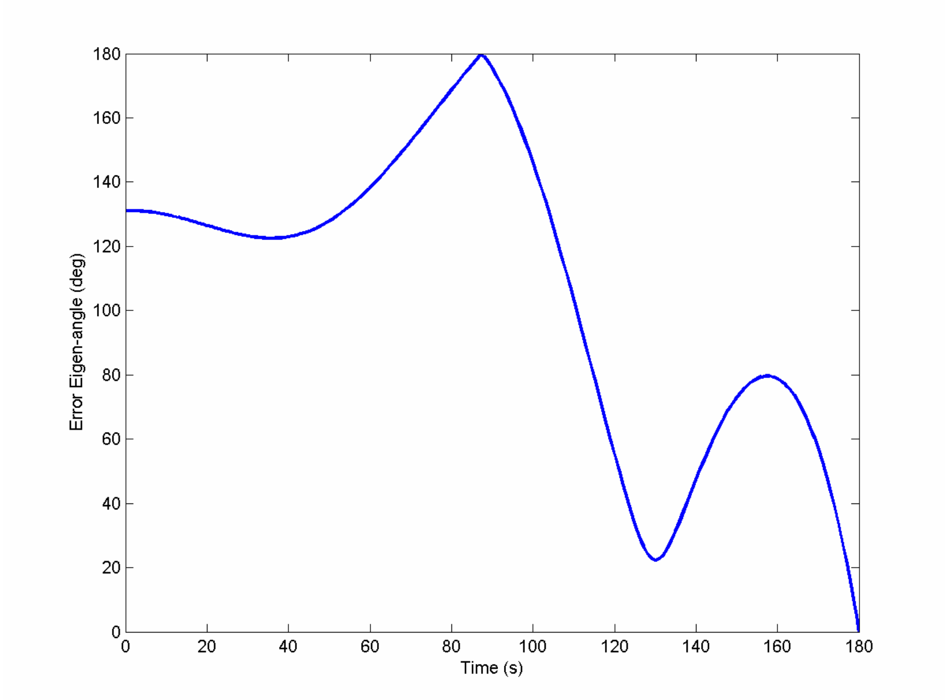

Figure 11 Agile maneuver error eigen-angle relative to target attitude

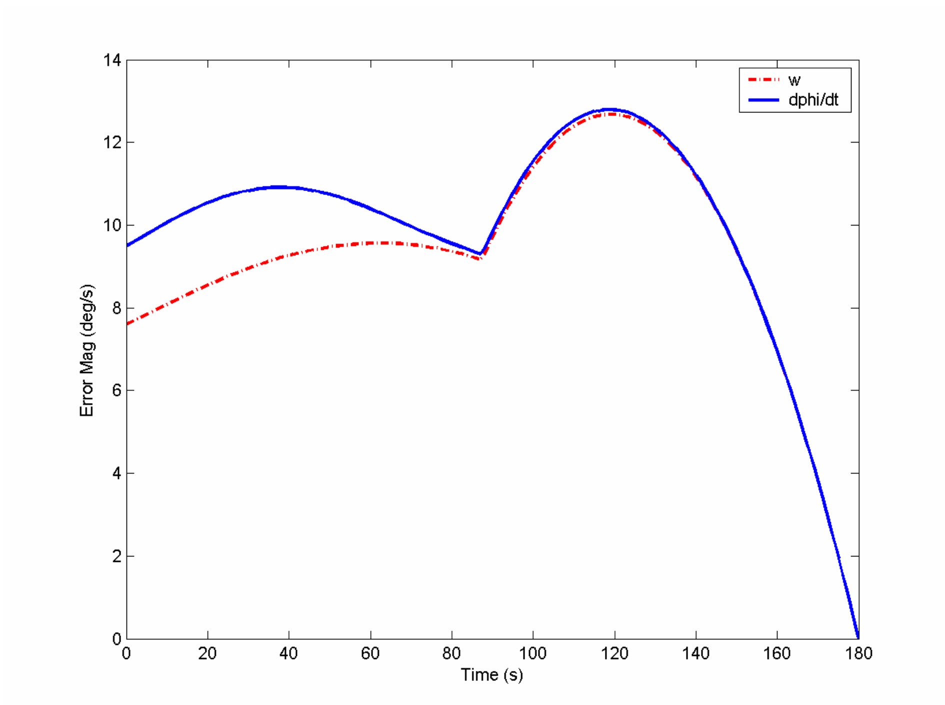

Figure 12 Agile maneuver error magnitudes in terms of angular velocity and rotation vector velocity relative to target attitude

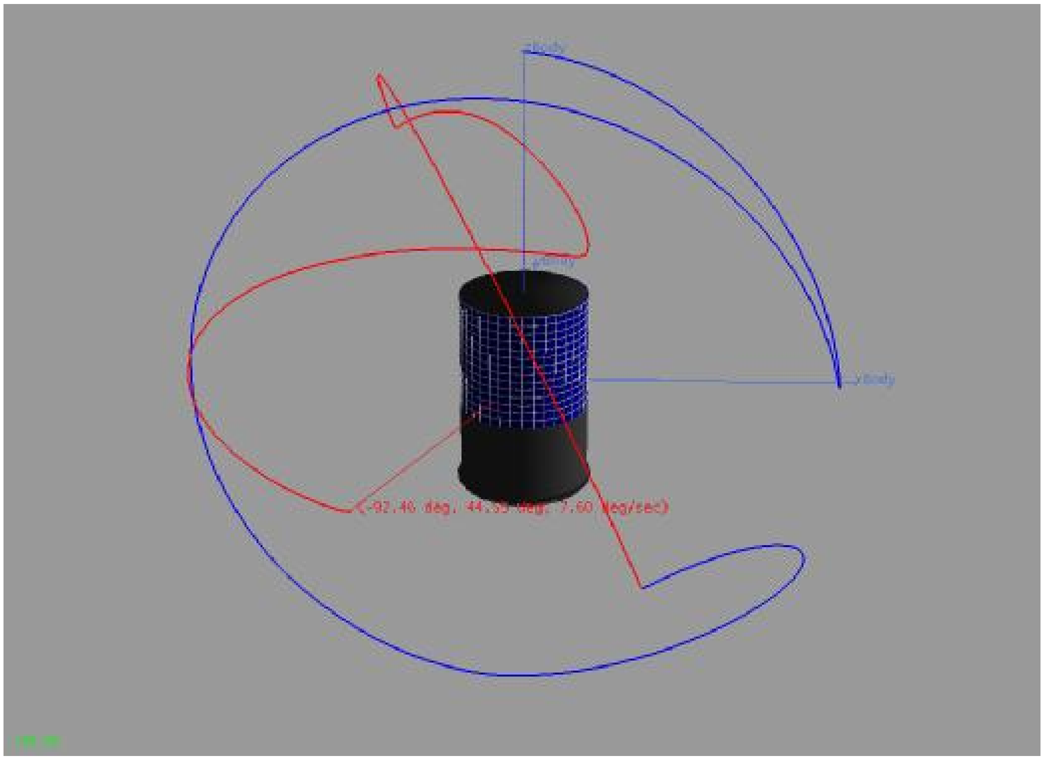

Figure 13 Agile maneuver motion of Z-body axis and angular velocity direction

CONCLUSIONS

The paper demonstrated that the attitude interpolation can be implemented using standard polynomial interpolation techniques and the rotation vector parameterization. The angular velocity data can be incorporated in the attitude interpolation improving the accuracy and removing possible aliasing of multiple revolutions.

The paper also demonstrated how the optimality of acceleration along 2-point osculating polynomials can be utilized in the design of near-optimal fixed duration maneuvers. The near-kinematically optimal closed-loop guidance law was designed based on successive and continuously updated interpolations.

The paper specifically addressed maneuver design for spinners, for which additional optimization is possible if the target clock-angle of the spin can be adjusted.

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